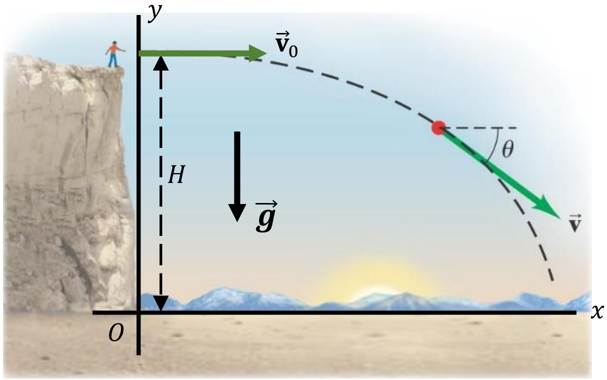
PHYS 101 Midterm Exam 1 Solution 2020-21-2

1. A ball is thrown horizontally from the top of a cliff with initial speed v_0. Height of the cliff from the ground below is H. Answer the following questions using the coordinate system indicated in the figure.



(a) (6 Pts.) What is the position vector of the ball as a function of time?
(b) (6 Pts.) Where does the ball hit the ground?
(c) (6 Pts.) What is the velocity of the ball at the instant it hits the ground?
(d) (9 Pts.) At any moment, its direction of motion makes an angle θ with the horizontal, as shown in the figure. Derive a formula for θ as a function of the horizontal position x of the ball.
(e) (8 Pts.) What is the velocity of the ball as a function of time relative to an observer moving with velocity $\vec{\mathbf{v}}_{O/G} = -v_0\,\hat{\mathbf{i}}$ with respect to the ground?

Solution: (a) Given $\vec{r}_0 = H\,\hat{\mathbf{j}}$, $\vec{v}_0 = v_0\,\hat{\mathbf{i}}$, and $\vec{a} = -g\,\hat{\mathbf{j}}$. Therefore,

$$\vec{r}(t) = (v_0 t)\,\hat{\mathbf{i}} + \left(H - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}, \qquad 0 \le t \le t_f\,.$$

(b) Ball hits the ground at time $t = t_f$ means $y(t_f) = 0$, or

$$H - \frac{1}{2}g{t_f}^2 = 0 \quad \rightarrow \quad t_f = \sqrt{\frac{2H}{g}}.$$

The x-coordinate of the ball at this instant is

$$x(t_f) = v_0 t_f = v_0\sqrt{\frac{2H}{g}}\,.$$

So, the point where the ball hits the ground is $\left(v_0\sqrt{\frac{2H}{g}},\ 0\right)$.

(c) The velocity of the ball at any time for $0 \le t \le t_f$ is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (v_0)\,\hat{\mathbf{i}} + (-gt)\,\hat{\mathbf{j}}\,.$$

Therefore,

$$\vec{v}(t_f) = (v_0)\,\hat{\mathbf{i}} - \left(\sqrt{2gH}\right)\hat{\mathbf{j}}\,.$$

(d)

$$\tan\theta = \frac{|v_y|}{v_x} = \frac{gt}{v_0} = \left(\frac{gx}{{v_0}^2}\right) \quad \rightarrow \quad \theta(x) = \arctan\left(\frac{gx}{{v_0}^2}\right).$$

(e)

$$\vec{v}_{B/O} = \vec{v}_{B/G} + \vec{v}_{G/O} = \vec{v}_{B/G} - \vec{v}_{O/G} \quad \rightarrow \quad \vec{v}_{B/O} = (2v_0)\,\hat{\mathbf{i}} + (-gt)\,\hat{\mathbf{j}}\,.$$

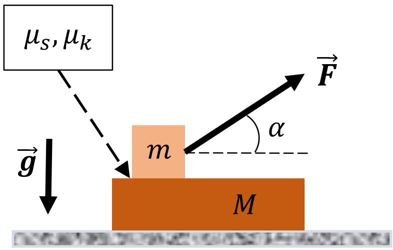
2. A block with mass m is placed on top of larger block with mass M. Coefficients of static and kinetic friction between the two blocks are μ_s and μ_k, respectively. Initially, both blocks are at rest on a frictionless horizontal surface. Starting at time $t = 0$, a constant force with magnitude F (assume: $F \sin\alpha < mg$) is applied to the top block at an angle α to the horizontal, as shown in the figure.

(a) (10 Pts.) Draw free-body diagrams of the two blocks for $t > 0$.

(b) (10 Pts.) What is the maximum value of F for which the top block remains at rest relative to the bottom block?

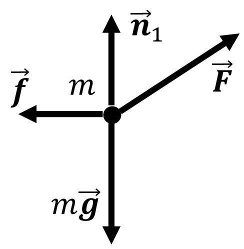
(c) (10 Pts.) If F is greater than the maximum value, what will be the acceleration of the top block relative to the bottom block?

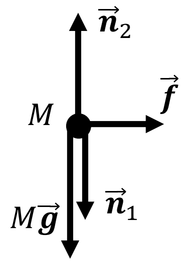
(d) (5 Pts.) What happens if $F \sin\alpha > mg$?



Solution:

(a)





(b) Top block does not slide on the bottom block, so the friction is static, and both blocks have the same horizontal acceleration a. From the free-body diagrams, we write

$F\cos\alpha - f_s = ma, n_1 + F\sin\alpha - mg = 0, f_s = Ma\,.$

So,

$$a = \frac{F\cos\alpha}{m+M}\,, \qquad f_s = \frac{M}{m+M}F\cos\alpha\,.$$

Since, for static friction we have $f_s \le \mu_s n_1$,

$$\frac{M}{m+M}F\cos\alpha \le \mu_s mg - \mu_s F\sin\alpha \quad \to \quad F_{\max} = \mu_s mg \Big/ \left(\frac{M}{m+M}\cos\alpha + \mu_s \sin\alpha\right).$$

(c) If F is greater than the maximum value found above, top block will be sliding on the bottom one. So the friction force will be kinetic, and the two blocks will have different horizontal accelerations. From the free-body diagrams, we write

$F\cos\alpha - f_k = ma_{\text{top}}, \qquad n_1 + F\sin\alpha - mg = 0, \qquad f_k = \mu_k n_1 = \mu_k(mg - F\sin\alpha)\,.$

So

$$a_{\text{top}} = \frac{F}{m}(\cos\alpha + \mu_k \sin\alpha) - \mu_k g\,.$$

Also,

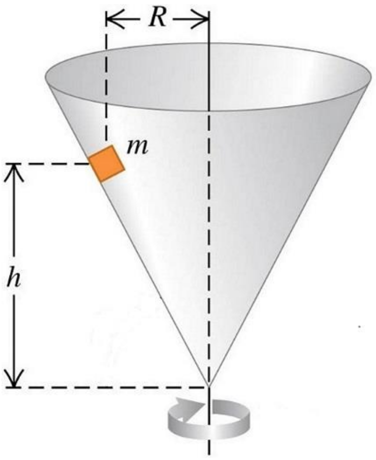
$$f_k = Ma_{\text{bot}} \quad \to \quad a_{\text{bot}} = \mu_k\left(\frac{m}{M}g - \frac{F}{M}\sin\alpha\right),$$

and

$$a_{\text{rel}} = a_{\text{top}} - a_{\text{bot}} = \frac{F}{m}\cos\alpha + \frac{m+M}{mM}\mu_k(F\sin\alpha - mg)\,.$$

(d) If $F\sin\alpha > mg$, then $n_1 = mg - F\sin\alpha < 0$, meaning that the top block is lifted off the lower block. Therefore, the bottom block does not accelerate, and the top block acceleraties in the direction of the net force, which is $\vec{F}_{\text{net}} = \vec{F} + (m\vec{g})$.

3. (30 Pts.) A small block with mass m is placed inside an inverted cone that is rotating about a vertical axis such that the time for one revolution of the cone is T. The coefficient of static friction between the block and the cone is μ_s. If the block is to remain at a constant height h above the apex of the cone, what are (a) the maximum value of T and (b) the minimum value of T in terms of g, μ_s, h, and R? (That is, find expressions for $T_{\max}$ and $T_{\min}$ in terms of g, μ_s, h, and R.)

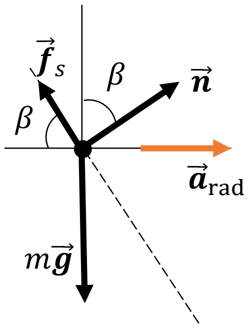


Solution:

Assume that the walls of the cone make an angle β with the horizontal. From the figure, we see that $\cos\beta = R/\sqrt{R^2+h^2}$, and $\sin\beta = h/\sqrt{R^2+h^2}$. The speed of the block is $v = 2\pi R/T$. If $v < v_{\min} = 2\pi R/T_{\max}$, the block will slide down the cone meaning that the friction force will be upward. So, we consider the top free-body diagram. Newton's second law is written as

$$n\sin\beta - f_s\cos\beta = ma_{\text{rad}}, \qquad n\cos\beta + f_s\sin\beta = mg.$$

Solving these equations for the unknowns n and f_s, we get



$$f_s = m(g\sin\beta - a_{\text{rad}}\cos\beta), \qquad n = m(g\cos\beta + a_{\text{rad}}\sin\beta).$$

or static friction, we have $f_s \le \mu_s n$. Therefore,

$$g\sin\beta - a_{\text{rad}}\cos\beta \le \mu_s g\cos\beta + \mu_s a_{\text{rad}}\sin\beta \quad\rightarrow\quad a_{\text{rad}} \ge \frac{g(\sin\beta - \mu_s\cos\beta)}{\cos\beta + \mu_s\sin\beta}.$$

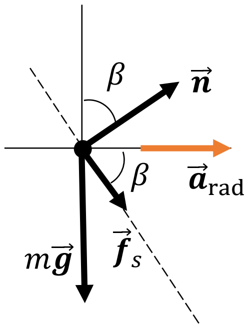
Since $a_{\text{rad}} = v^2/R$,

$$\frac{v^2}{R} \ge \frac{g(h-\mu_s R)}{R+\mu_s h} \quad\rightarrow\quad v_{\min} = \sqrt{Rg\left(\frac{h-\mu_s R}{R+\mu_s h}\right)},$$

and the result is found as

$$v_{\min} = 2\pi R/T_{\max} \quad\rightarrow\quad T_{\max} = 2\pi\sqrt{\frac{R}{g}\left(\frac{R+\mu_s h}{h-\mu_s R}\right)}.$$

If $v > v_{\max} = 2\pi R/T_{\min}$, the block will slide up the cone meaning that the friction force will be downward. So, we consider the second free-body diagram. Following the same steps, we have



$$n\sin\beta + f_s\cos\beta = ma_{\text{rad}}, \qquad n\cos\beta - f_s\sin\beta = mg.$$

Solving these equations for the unknowns n and f_s, we get

$$f_s = m(-g\sin\beta + a_{\text{rad}}\cos\beta), \qquad n = m(g\cos\beta + a_{\text{rad}}\sin\beta).$$

$$f_s \le \mu_s n \quad\rightarrow\quad -g\sin\beta + a_{\text{rad}}\cos\beta \le \mu_s g\cos\beta + \mu_s a_{\text{rad}}\sin\beta \quad\rightarrow\quad a_{\text{rad}} \le \frac{g(\sin\beta + \mu_s\cos\beta)}{\cos\beta - \mu_s\sin\beta}.$$

$$\frac{v^2}{R} \le \frac{g(\sin\beta + \mu_s\cos\beta)}{\cos\beta - \mu_s\sin\beta} \quad\rightarrow\quad v_{\max} = \sqrt{Rg\left(\frac{\sin\beta + \mu_s\cos\beta}{\cos\beta - \mu_s\sin\beta}\right)}.$$

So

$$T_{\min} = 2\pi\sqrt{\frac{R}{g}\left(\frac{R-\mu_s h}{h+\mu_s R}\right)}.$$